VOLUME GROWTH, EIGENVALUE AND COMPACTNESS FOR SELF-SHRINKERS

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Abstract. In this paper, we show an optimal volume growth for self-shrinkers, and estimate a lower bound of the first eigenvalue of $\mathcal{L}$ operator on self-shrinkers, inspired by the first eigenvalue conjecture on minimal hypersurfaces in the unit sphere. By the eigenvalue estimates, we can prove a compactness theorem obtained by Colding-Minicozzi under weaker conditions.

1. Introduction

Let $X : M^n \to \mathbb{R}^{n+m}$ be an isometric immersion from an n-dimensional manifold M^n to Euclidean space $\mathbb{R}^{n+m}(m \geq 1)$ with tangent bundle TM and normal bundle NM along M. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on M and $\mathbb{R}^{n+m}$, respectively. Then we define the second fundamental form B by, $B(V,W) = (\overline{\nabla}_V W)^N = \overline{\nabla}_V W - \nabla_V W$ for any $V, W \in \Gamma(TM)$, where $(\cdots)^N$ denotes the orthogonal projection into the normal bundle NM. The mean curvature vector H of M is given by $H = \mathrm{trace}(B) = \sum_{i=1}^n B(e_i, e_i)$, where $\{e_i\}$ is a local orthonormal frame field of M.

M^n is said to be a *self-shrinker* in $\mathbb{R}^{n+m}$ if it satisfies

$$H = -\frac{X^N}{2}. \tag{1.1}$$

Here, the factor $-\frac{1}{2}$ (when the codimension $m = 1$, the definition here is as the same as [4]) could be replaced by other negative number, while Ecker-Huisken defines $H = -X^N$ [8]. Self-shrinkers play an important role in the study of mean curvature flow. They are not only the simplest solution to the mean curvature flow equations (those where later time slices are rescalings of earlier), but they also describe all possible blow ups at a given singularity of a mean curvature flow (abbreviated by MCF in what follows).

After the pioneer work on self-shrinking hypersurfaces of G. Huisken [10][11], T. H. Colding and W. P. Minicozzi II gave a comprehensive study for self-shrinking hypersurfaces

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[4]. Their papers reveal the importance of the subject. For higher codimension, there is a few study, see [15] for example.

There are several other ways to characterize self-shrinkers that are equivalent to the equation (1.1)(see [5] for hypersurface, and high codimension is similar):
(1) The one-parameter family of submanifolds $\sqrt{-t}M \subset \mathbb{R}^{n+m}$ satisfy MCF equations.
(2) M is a minimal submanifold in $\mathbb{R}^{n+m}$ endowed with the conformally flat metric of the conformal factor $e^{-\frac{|X|^2}{2n}}$.
(3) M is a critical point for the functional F defined on a submanifold $M \subset \mathbb{R}^{n+m}$ by

$$F(M) = (4\pi)^{-n/2} \int_M e^{-\frac{|X|^2}{4}} d\mu. \tag{1.2}$$

Self-shrinkers satisfy elliptic equations of the second order, see (1.1). It is an important class of submanifolds, which is closely related to minimal surface theory. We expect certain technique in minimal surface theory (see [17]) could be modified to study self-shrinkers.

For a complete non-compact manifold the volume growth is important. By easy arguments we can show that any complete self-shrinker in Euclidean space with arbitrary codimension has Euclidean volume growth, just like the trivial self-shrinker: planes. It is in a sharp contrast to the complete minimal submanifolds in Euclidean space. Even for complete stable minimal hypersurfaces, it is still unclear whether they have Euclidean volume growth.

Theorem 1.1. *Any complete non-compact properly immersed self-shrinker M^n in $\mathbb{R}^{n+m}$ has Euclidean volume growth at most.*

It is natural to raise a counterpart of the the Calabi-Chern problem on minimal surfaces in $\mathbb{R}^3$. Is there a complete non-compact self-shrinker in Euclidean space, which is contained in a Euclidean ball?

Remark 1.2. *It is worthy to compare the above Theorem with the interesting result of Cao-Zhou on the volume growth of complete gradient shrinking Ricci soliton* [3].

Let Σ^n be a compact embedded minimal hypersurface in $n+1$-dimensional sphere $\mathbb{S}^{n+1}$. It is well known that the coordinate functions are eigenfunctions of Laplacian operator on Σ with eigenvalue n. In [14], S. T. Yau conjectured that the first eigenvalue of Σ would be n. Choi-Wang, [2], proved that the first eigenvalue of Σ is bounded below by $n/2$.

In [1], H. I. Choi and R. Schoen gave the compactness theorem for minimal surfaces using the first eigenvalue estimates for Laplacian operator on Σ. Precisely, let N be a compact 3-dimensional manifold with positive Ricci curvature, then the space of compact embedded minimal surfaces of fixed topological type in N is compact in the C^k topology for any $k \geq 2$.

In [5], Colding-Minicozzi proved a compactness theorem for complete embedded self shrinkers in $\mathbb{R}^3$. Such compactness theorem acts a key role for proving a long-standing conjecture (of Huisken) classifying the singularities of mean curvature flow starting at a generic closed embedded surface (see [4]).

Let Δ, div and $d\mu$ are Laplacian, divergence and volume element on M, respectively. There is a linear operator

$$\mathcal{L} = \Delta - \frac{1}{2}\langle X, \nabla(\cdot)\rangle = e^{\frac{|X|^2}{4}}\mathrm{div}(e^{-\frac{|X|^2}{4}}\nabla\cdot).$$

On Euclidean space, this operator is so-called Ornstein-Uhlenbeck operator in stochastic analysis. So it can be seen as a generalization of Ornstein-Uhlenbeck operator. The $\mathcal{L}$ operator was introduced and studied firstly on self-shrinkers by Colding-Minicozzi in [4], where the authors also showed that $\mathcal{L}$ is self-adjoint respect to measure $e^{-\frac{|X|^2}{4}}d\mu$. It is a weighted Laplacian and closely related to the self-shrinkers.

In Euclidean space the eigenvalues of the Ornstein-Uhlenbeck is well-known. On self-shrinkers it is interesting to study the eigenvalues of $\mathcal{L}$ operator. Now, we estimate its first eigenvalue in a manner analogous to the arguments in [2]. For compact self-shrinkers the estimates is rather neat. It is also enough for the compactness applications. In fact, we could do it on a class of complete self-shrinkers. Now we give the definition of this type of manifolds.

We say that an embedded hypersurface Σ in $\mathbb{R}^{n+1}$ (with or without boundary) satisfies *two sides δ_0-ball condition*, if there is a positive constant δ_0, for any point $X \in \Sigma$, there are two different open balls $B_1, B_2 \subset \mathbb{R}^{n+1}$ with radius $\delta_0 \cdot \min\{1, 1/|X|^2\}$ in $\mathbb{R}^{n+1}$ such that $\overline{B_1} \cap \Sigma = X$ and $\overline{B_2} \cap \Sigma = X$.

In general, a sequence of complete self-shrinkers $\{M_i\}$ have not uniformly Euclidean volume growth where M_i satisfies two sides δ_i-ball condition for some $\delta_i > 0$, as we don't require the sequence $\{\delta_i\}$ has a uniformly lower bound. And any compact self-shrinker must satisfy two sides δ_0-ball condition for some $\delta_0 > 0$.

Theorem 1.3. *Let M^n is a complete embedded self-shrinker with two sides δ_0-ball condition, then the first eigenvalue λ_1 for the operator $\mathcal{L}$ on M satisfying: $\lambda_1 \in [\frac{1}{4}, \frac{1}{2}]$.*

By using this estimates we can estimate uniform volume growth for compact embedded self-shrinkers with bounded genus and can get a compactness theorem. Given a non-negative integer g and a constant $D > 0$, we let $S_{g,D}$ denote the space of all compact embedded self-shrinkers in $\mathbb{R}^3$ with genus at most g, and diameter at most D. We have a compactness theorem as follows.

Theorem 1.4. *For each fixed g and D, the space $S_{g,D}$ is compact. Namely, any sequence has a subsequence that converges uniformly in the C^k topology (for any $k \geq 0$) to a surface in $S_{g,D}$.*

In Corollary 8.2 of [4], Colding and Minicozzi II proved the above theorem under bounded entropy, which is natural appeared in their Theorem 0.10 in [4].

In this paper, we always suppose that M is a smooth submanifold of the dimension $n \geq 2$, the function $\rho = e^{-\frac{|X|^2}{4}}$, $\langle \cdot, \cdot \rangle$ is standard inner product of $\mathbb{R}^{n+m}$, and B_r is a standard ball in $\mathbb{R}^{n+m}$ with radius r and centered at the origin and $D_r = M \cap B_r$ for the submanifold M in $\mathbb{R}^{n+m}$. When $m = 1$ (codimension is 1), let ν be unit outward normal field of M, and $\langle H, \nu \rangle$ be mean curvature of M. We also write $H = \langle H, \nu \rangle$, which will not arise confusion in the context. We agree with the following range of indices

$$1 \leq i, j, k, \cdots \leq n + m, \quad 1 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

2. Volume Growth of Self Shrinkers

Let M be an n-dimensional complete self shrinkers in $\mathbb{R}^{n+m}$, by (1.1), we have $\Delta X = H = -\frac{1}{2} X^N$ for any $X = (x_1, \cdots, x_{n+m}) \in \mathbb{R}^{n+m}$, then(see also [4])

$$\mathcal{L} x_i = \Delta x_i - \frac{1}{2}\langle X, \nabla x_i \rangle = -\frac{1}{2}\langle X^N, E_i \rangle - \frac{1}{2}\langle X, (E_i)^T \rangle = -\frac{1}{2} x_i, \tag{2.1}$$

where $\{E_i\}_{i=1}^{n+m}$ is a standard basis of $\mathbb{R}^{n+m}$ and $(\cdots)^T$ denote orthogonal projection into the tangent bundle TM. Moreover,

$$\mathcal{L}|X|^2 = 2x_i \mathcal{L} x_i + 2|\nabla X|^2 = 2n - |X|^2, \tag{2.2}$$

and

$$\Delta|X|^2 = 2\langle X, \Delta X\rangle + 2|\nabla X|^2 = 2\langle X, H\rangle + 2n = 2n - 4|H|^2. \tag{2.3}$$

Now, we give an analytic lemma which will be used in proving volume growth.

Lemma 2.1. *If $f(r)$ is a monotonic increasing nonnegative function on $[0, +\infty)$ satisfying $f(r) \leq C_1 r^n f(\frac{r}{2})$ on $[C_2, +\infty)$ for some positive constant n, C_1, C_2, here $C_2 > 1$, then $f(r) \leq C_3 e^{2n(\log r)^2}$ on $[C_2, +\infty)$ for some positive constant C_3 depending only on $n, C_1, C_2, f(C_2)$.*

Proof. If $f(\frac{C_2}{2}) = 0$, then $f(r) = 0$ for $r \geq \frac{C_2}{2}$. So we assume $f(\frac{C_2}{2}) > 0$, then $g(r) = \log f(r)$ on $[C_2, \infty)$ is well defined. By the assumption, on $[C_2, +\infty)$

$$g(r) \leq g(\frac{r}{2}) + \log C_1 + n \log r.$$

Let $k = [\frac{\log \frac{r}{C_2}}{\log 2}] + 1$, then $\frac{C_2}{2} \leq \frac{r}{2^k} < C_2$, which implies $\frac{r}{2^{k-1}} \geq C_2 > 1$. By iteration,

$$\begin{aligned} g(r) \leq& g(\frac{r}{2^2}) + 2\log C_1 + n(\log r + \log \frac{r}{2}) \leq \cdots \\ \leq& g(\frac{r}{2^k}) + k \log C_1 + n \sum_{j=0}^{k-1} \log \frac{r}{2^j}. \end{aligned} \tag{2.4}$$

From (2.4) we have

$$\begin{aligned} g(r) \leq& g(C_2) + k(\log C_1 + n \log r) \\ \leq& g(C_2) + (\frac{\log \frac{r}{C_2}}{\log 2} + 1)(\log C_1 + n \log r) \\ \leq& \log C_3 + 2n(\log r)^2, \end{aligned}$$

for some positive constant C_3 depending only on $n, C_1, C_2, f(C_2)$, hence by the definition of g, we have

$$f(r) \leq C_3 e^{2n(\log r)^2}$$

on $[C_2, +\infty)$. ∎

For a complete non-compact $n-$submanifold M in $\mathbb{R}^{n+m}$, we say that M has *Euclidean volume growth at most* if there is a constant C so that for all $r \geq 1$,

$$\int_{D_r} 1 d\mu \leq C r^n.$$

For a complete self-shrinker M^n in $\mathbb{R}^{n+m}$, we define a functional F_t on any set $\Omega \subset M$ (see also [4] for the definition of F_t) by,

$$F_t(\Omega) = \frac{1}{(4\pi t)^{n/2}} \int_\Omega e^{-\frac{|X|^2}{4t}} d\mu, \quad \text{for} \quad t > 0.$$

Theorem 2.2. *Any complete non-compact properly immersed self-shrinker M^n in $\mathbb{R}^{n+m}$ has Euclidean volume growth at most. Precisely, $\int_{D_r} 1 d\mu \le Cr^n$ for $r \ge 1$, where C is a constant depending only on n and the volume of D_{8n}.*

Proof. We differential $F_t(D_r)$ with respect to t,

$$F_t'(D_r) = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_{D_r} (-\frac{n}{2} + \frac{|X|^2}{4t}) e^{-\frac{|X|^2}{4t}} d\mu.$$

Since

$$\begin{aligned} -e^{\frac{|X|^2}{4t}} \mathrm{div}(e^{-\frac{|X|^2}{4t}} \nabla |X|^2) &= -\Delta |X|^2 + \frac{1}{4t} \nabla |X|^2 \cdot \nabla |X|^2 \\ &= -2\langle H, X \rangle - 2n + \frac{1}{4t} 4 |X^T|^2 \\ &= |X^N|^2 + \frac{|X^T|^2}{t} - 2n \\ &\ge \frac{|X|^2}{t} - 2n \quad (\ when\ t \ge 1\ ), \end{aligned} \tag{2.5}$$

where in the third equality above the self-shrinker's equation (1.1) is used. Since

$$\nabla |X|^2 = 2X^T$$

and the unit normal vector to ∂D_r is $\frac{X^T}{|X^T|}$, then for $t \ge 1$,

$$\begin{aligned} F_t'(D_r) \le& \pi^{-\frac{n}{2}} (4t)^{-(\frac{n}{2}+1)} \int_{D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}} \nabla |X|^2) d\mu \\ =& \pi^{-\frac{n}{2}} (4t)^{-(\frac{n}{2}+1)} \int_{\partial D_r} -2|X^T| e^{-\frac{|X|^2}{4t}} \le 0. \end{aligned} \tag{2.6}$$

By (2.6), we integral $F_t'(D_r)$ over t from 1 to $r^2 \ge 1$ and get

$$(4\pi r^2)^{-\frac{n}{2}} \int_{D_r} e^{-\frac{|X|^2}{4r^2}} d\mu \le (4\pi)^{-\frac{n}{2}} \int_{D_r} e^{-\frac{|X|^2}{4}} d\mu.$$

Then

$$\begin{aligned} \frac{1}{r^n} e^{-\frac{1}{4}} \int_{D_r} 1 d\mu \le& \frac{1}{r^n} \int_{D_r} e^{-\frac{|X|^2}{4r^2}} d\mu \le \int_{D_r} e^{-\frac{|X|^2}{4}} d\mu = \int_{(D_r \setminus D_{\frac{r}{2}}) \cup D_{\frac{r}{2}}} e^{-\frac{|X|^2}{4}} d\mu \\ \le& e^{-\frac{r^2}{16}} \int_{D_r} 1 d\mu + \int_{D_{\frac{r}{2}}} 1 d\mu. \end{aligned} \tag{2.7}$$

Let $f(r)=\int_{D_r} 1 d\mu$, then by (2.7),

$$f(r)\leq 2e^{\frac{1}{4}}r^n f(\frac{r}{2}) \quad for \quad r\geq 8n.$$

By Lemma 2.1, we have

$$f(r)\leq C_4 e^{2n(\log r)^2} \quad for \quad r\geq 8n,$$

here C_4 is a constant depending only on $n, f(8n)$. Hence

$$\begin{aligned}\int_M e^{-\frac{|X|^2}{4}}d\mu =&\sum_{j=0}^{\infty}\int_{D_{8n(j+1)}\setminus D_{8nj}} e^{-\frac{|X|^2}{4}}d\mu \leq \sum_{j=0}^{\infty} e^{-\frac{(8nj)^2}{4}} f(8n(j+1))\\ \leq& C_4\sum_{j=0}^{\infty} e^{-\frac{(8nj)^2}{4}} e^{2n(\log(8n)+\log(j+1))^2}\leq C_5,\end{aligned}$$

where C_5 is a constant depending only on $n, f(8n)$. By (2.7), we finish the proof. □

Let M be a complete embedded self shrinking hypersurface in $\mathbb{R}^{n+1}$ with nonpositive mean curvature, namely, $H\leq 0$, then $\langle X,\nu\rangle\geq 0$ and M is proper. By the Theorem 2.2, M has Euclidean volume growth. By Huisken's classification theorem(see [4], [10] and [11]), we obtain that the only smooth complete embedded self-shrinkers in $\mathbb{R}^{n+1}$ with mean curvature $H\leq 0$ are isometric to $\mathbb{S}^k\times\mathbb{R}^{n-k}$ for $0\leq k\leq n$.

If M is an entire graphic self shrinking hypersurface in $\mathbb{R}^{n+1}$, then M has Euclidean volume growth. Then by [8], we know that M is a hyperplane, which is also obtained by Lu Wang [16].

Remark 2.3. *Let M^n be a complete properly immersed self-shrinker in $\mathbb{R}^{n+m}$, by (2.2), then(see also [4])*

$$\int_M |X|^2\rho = 2n\int_M \rho. \tag{2.8}$$

For any $0<\epsilon<\sqrt{2n}$,

$$2\sqrt{2n}\epsilon\int_{M\setminus D_{\sqrt{2n}+\epsilon}}\rho\leq\int_{M\setminus D_{\sqrt{2n}+\epsilon}}(|X|^2-2n)\rho=\int_{D_{\sqrt{2n}+\epsilon}}(2n-|X|^2)\rho\leq 2n\int_{D_{\sqrt{2n}}}\rho.$$

Let η is a cut-off function satisfying $\eta\mid_{B_{\sqrt{2n}-\epsilon}}\equiv 1$, $\eta\mid_{\mathbb{R}^{n+m}\setminus B_{\sqrt{2n}}}\equiv 0$ and $|\nabla\eta|\leq\frac{1}{\epsilon}$, then

$$\sqrt{2n}\epsilon\int_{D_{\sqrt{2n}-\epsilon}}\rho\leq\int_{D_{\sqrt{2n}}}(2n-|X|^2)\eta\rho=2\int_{D_{\sqrt{2n}}}\nabla\eta\cdot\frac{X^T}{|X|}\rho \quad\leq\frac{2}{\epsilon}\int_{D_{\sqrt{2n}}\setminus D_{\sqrt{2n}-\epsilon}}\rho.$$

Hence, we conclude that there is a constant C_6, C_7 depending only on n and ϵ, such that

$$\int_M \rho d\mu\leq C_6\int_{M\cap(B_{\sqrt{2n}+\epsilon}\setminus B_{\sqrt{2n}-\epsilon})}\rho d\mu. \tag{2.9}$$

Combining (2.7) *and* (2.9), *we have*

$$\int_{D_r} 1d\mu \leq C_7 r^n \int_{M\cap(B_{\sqrt{2n}+\epsilon}\setminus B_{\sqrt{2n}-\epsilon})} 1d\mu.$$

Corollary 2.4. *Let M^n be a complete non-compact properly immersed self-shrinker in $\mathbb{R}^{n+m}$, then $F_t(M)\leq F_1(M)$ for any $t>0$.*

Proof. Let ζ is a cut-off function such that

$$\zeta(|X|)=\begin{cases} 1 & \text{if} \quad X\in B_r \\ linear & \text{if} \quad X\in B_{2r}\setminus B_r \\ 0 & \text{if} \quad X\in\mathbb{R}^{n+m}\setminus B_{2r}, \end{cases}$$

Combining (2.5), for any $t>0$,

$$\begin{aligned} &\left|\int_{D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu\right| \\ \leq&\left|\int_{M} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)\zeta d\mu\right| + \left|\int_{M\setminus D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)\zeta d\mu\right| \\ \leq&\left|\int_{M} (\nabla|X|^2\cdot\nabla\zeta)e^{-\frac{|X|^2}{4t}}d\mu\right| + \int_{M\setminus D_r}\left||X^N|^2+\frac{|X^T|^2}{t}-2n\right|e^{-\frac{|X|^2}{4t}}d\mu \\ \leq&\frac{2}{r}\int_M |X|e^{-\frac{|X|^2}{4t}}d\mu + \int_{M\setminus D_r}\left((1+\frac{1}{t})|X|^2+2n\right)e^{-\frac{|X|^2}{4t}}d\mu, \end{aligned}$$

which implies

$$\lim_{r\to\infty}\int_{D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu=0. \tag{2.10}$$

When $0<t\leq 1$, from (2.5), we have

$$-e^{\frac{|X|^2}{4t}}\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla_M|X|^2)\leq\frac{|X|^2}{t}-2n,$$

then

$$F_t'(D_r)\geq\pi^{-\frac{n}{2}}(4t)^{-(\frac{n}{2}+1)}\int_{D_r} -\mathrm{div}(e^{-\frac{|X|^2}{4t}}\nabla|X|^2)d\mu. \tag{2.11}$$

Combining (2.10) and (2.11), we know

$$F_t'(M)=\lim_{r\to\infty}F_t'(D_r)\geq 0,$$

which implies

$$F_t(M)\leq F_1(M), \quad for\ 0<t\leq 1.$$

On the other hand, by (2.6), we have

$$(4\pi R^2)^{-n/2}\int_{D_r} e^{-\frac{|X|^2}{4R^2}}d\mu \le (4\pi)^{-n/2}\int_{D_r} e^{-\frac{|X|^2}{4}}d\mu \qquad for\ any \quad R\ge 1.$$

Let r go to infinite, thus we finish the proof. □

3. The First Eigenvalue of Self-shrinkers and Compactness Theorem

Let $\mathbb{R}^{n+1}$ be Euclidean space with canonical metric, whose Levi-Civita connection denoted by $\overline{\nabla}$, Laplacian denoted by $\overline{\Delta}$, divergence denoted by $\overline{\mathrm{div}}$. Let $\overline{\mathcal{L}} = \overline{\Delta} - \frac{1}{2}\langle X, \overline{\nabla}\cdot\rangle$. Reilly derive a useful integral formula on Laplace operator [13](see also [2]). Now, we derive a Reilly type formula for the operator $\mathcal{L}$.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with smooth boundary. Suppose f is a function defined in Ω satisfying*

$$\begin{cases} \overline{\mathcal{L}}f = g & \text{in} \quad \Omega \\ f = u & \text{on} \quad \partial\Omega, \end{cases}$$

then

$$\int_\Omega g^2\rho = \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + 2\int_{\partial\Omega} f_\nu \mathcal{L}u\rho - \int_{\partial\Omega} h(\nabla u, \nabla u)\rho - \int_{\partial\Omega} f_\nu^2(\frac{\langle X,\nu\rangle}{2} + H)\rho,$$

where $h(\cdot,\cdot) = \langle B(\cdot,\cdot),\nu\rangle$, $B(\cdot,\cdot)$ is the second fundamental form on $\partial\Omega$, ν is outward unit normal vector field on $\partial\Omega$ and mean curvatue $H = \mathrm{trace}(h)$.

Proof. Let $\{\frac{\partial}{\partial x_i}\}_{i=1}^{n+1}$ be a canonical basis of $\mathbb{R}^{n+1}$, $f_i = \frac{\partial f}{\partial x_i}$, and so on. By $\overline{\mathcal{L}}f = g$, we have

$$\overline{\mathcal{L}}f_i = \sum_j (f_{ijj} - \frac{1}{2}x_j f_{ij}) = \frac{\partial}{\partial x_i}(g + \frac{1}{2}\sum_j x_j f_j) - \frac{1}{2}\sum_j x_j f_{ij} = g_i + \frac{f_i}{2},$$

and

$$\frac{1}{2}\overline{\mathcal{L}}|\overline{\nabla} f|^2 = \sum_{i,j} f_{ij}^2 + f_i\overline{\mathcal{L}}f_i = |\overline{\nabla}^2 f|^2 + \langle\overline{\nabla} f, \overline{\nabla} g\rangle + \frac{1}{2}|\overline{\nabla} f|^2. \tag{3.1}$$

Integrating the equality (3.1) by parts we get

$$\begin{aligned} \frac{1}{2}\int_\Omega \overline{\mathcal{L}}|\overline{\nabla} f|^2\rho &= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \int_\Omega \langle\overline{\nabla} f, \overline{\nabla} g\rangle\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho \\ &= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + \int_\Omega (\overline{\mathrm{div}}(\rho g\overline{\nabla} f) - g\overline{\mathrm{div}}(\rho\overline{\nabla} f)) \\ &= \int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + \int_{\partial\Omega} f_\nu g\rho - \int_\Omega g^2\rho. \end{aligned} \tag{3.2}$$

On the other hand, choose an orthonormal frame field $\{e_1,\cdots,e_{n+1}\}$ near the boundary of Ω such that $\{e_1,\cdots,e_n\}$ are tangential to $\partial\Omega$, and $\nabla_{e_\alpha}e_\beta=\overline{\nabla}_{e_{n+1}}e_i=0$ at a considered point in $\partial\Omega$ and $\nu=e_{n+1}$ is the outward unit normal vector, let $h_{\alpha\beta}=\langle\overline{\nabla}_{e_\alpha}e_\beta,\nu\rangle=\langle B(e_\alpha,e_\beta),\nu\rangle$. Then integrating by parts

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\overline{\mathcal{L}}|\overline{\nabla}f|^2\rho=\frac{1}{2}\int_\Omega\overline{\mathrm{div}}(\rho\overline{\nabla}|\overline{\nabla}f|^2)=\int_{\partial\Omega}\sum_{i=1}^{n+1}(e_if)(e_{n+1}e_if)\rho\\
=&\int_{\partial\Omega}f_\nu(e_{n+1}e_{n+1}f)\rho+\sum_{\alpha=1}^n\int_{\partial\Omega}(e_\alpha f)(e_\alpha e_{n+1}f)\rho+\sum_{\alpha=1}^n\int_{\partial\Omega}[e_{n+1},e_\alpha](f)(e_\alpha f)\rho\\
=&\int_{\partial\Omega}f_\nu(e_{n+1}e_{n+1}f)\rho-\int_{\partial\Omega}f_\nu(\mathcal{L}u)\rho+\sum_{\alpha,\beta=1}^n\int_{\partial\Omega}h_{\alpha\beta}e_\beta(f)e_\alpha(f)\rho.
\end{aligned}
\tag{3.3}
$$

Moreover,

$$
\begin{aligned}
e_{n+1}e_{n+1}f=&\sum_{i=1}^{n+1}\left(e_ie_if-\left(\overline{\nabla}_{e_i}e_i\right)f\right)-\sum_{\alpha=1}^n(e_\alpha e_\alpha f)+\sum_{\alpha=1}^n\left(\overline{\nabla}_{e_\alpha}e_\alpha\right)f\\
=&\overline{\Delta}f-\Delta f+\sum_{\alpha=1}^n h_{\alpha\alpha}f_\nu=\overline{\mathcal{L}}f-\mathcal{L}u+\frac{\langle X,\nu\rangle}{2}f_\nu+Hf_\nu.
\end{aligned}
\tag{3.4}
$$

Combining (3.2)-(3.4), we complete the proof. $\square$

We would use the above Reilly type formula to estimate the first eigenvalue of $\mathcal{L}$ operator on a self-shrinker in Euclidean space. Now the ambient space is not compact. We need the following boundary gradient estimate for $\overline{\mathcal{L}}$.

Lemma 3.2. *Let Σ is an n-dimensional compact embedded manifold in $\mathbb{R}^{n+1}$, Ω be a bounded domain in $\mathbb{R}^{n+1}$ with $\partial\Omega=\Sigma\cup S_R$. Here S_R is n-sphere with radius R and centered at the origin for any $R\geq\sqrt{2(n+1)}+diam(\Sigma)$. We consider Dirichlet problem*

$$
\begin{cases}
\overline{\mathcal{L}}f=0 & \text{in}\quad\Omega\\
f\mid_\Sigma=u\ ,\ f\mid_{S_R}=0\ ,
\end{cases}
$$

where u is a smooth function on Σ, then $|\overline{\nabla}f(X_0)|\leq 3\max_{X\in\Sigma}|u(X)|R$ for any $X_0\in S_R$.

Proof. For any $X_0\in S_R$, there is a unique $Y_0\in\mathbb{R}^{n+1}$ such that $\overline{B_R(0)}\cap\overline{B_R(Y_0)}=X_0$. Let $u_0=\max_{X\in\Sigma}|u(X)|$ and define two barrier functions $w^\pm(d)=\pm3u_0(1-\exp(-\frac{d^2-R^2}{2}))$, $d(X)=|X-Y_0|$ on the ball $B_{\sqrt{R^2+1}}(Y_0)$. Now, we prove that the two functions $w^\pm$ satisfy

(i) $\quad\pm\overline{\mathcal{L}}w^\pm<0\qquad$ in $B_{\sqrt{R^2+1}}(Y_0)\cap\Omega$,

(ii) $\quad w^\pm(X_0)=f(X_0)=0$,

(iii) $\quad w^-(X) \leq f(X) \leq w^+(X), \quad X \in \partial B_{\sqrt{R^2+1}}(Y_0) \cap \Omega.$

Let $Y = X - Y_0$, then $d = |Y|$, $\overline{\nabla} d = \frac{Y}{|Y|}$ and $\overline{\Delta} d = \frac{n}{|Y|}$, hence

$$\overline{\mathcal{L}} w^+ = (w^+)' \overline{\mathcal{L}} d + (w^+)'' |\overline{\nabla} d|^2 = (w^+)'' + (w^+)' (\frac{n}{|Y|} - \frac{1}{2} \frac{X \cdot Y}{|Y|}).$$

Since $(w^+)' = 3u_0 d e^{-\frac{d^2-R^2}{2}}$ and $(w^+)'' = 3u_0(1-d^2) e^{-\frac{d^2-R^2}{2}}$, then for any $X \in B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$, we have $|X| \leq R$, $d = |Y| \geq R$ and

$$\begin{aligned} \overline{\mathcal{L}} w^+ \leq & 3u_0(1-d^2) e^{-\frac{d^2-R^2}{2}} + 3u_0 d e^{-\frac{d^2-R^2}{2}} (\frac{n}{d} + \frac{R}{2}) \\ = & 3u_0 e^{-\frac{d^2-R^2}{2}} (1 - d^2 + n + \frac{R}{2} d) \leq 0 \quad (\textit{since } R \geq \sqrt{2(n+1)}\ ), \end{aligned}$$

(i) is proved. (ii) is obviously. By maximum principle, we have

$$|f(X)| \leq u_0, \quad \text{for any } X \in \Omega.$$

When $X \in \partial B_{\sqrt{R^2+1}}(Y_0) \cap \Omega$, $w^+(X) = 3u_0(1 - e^{-1/2}) \geq f(X)$ and $w^-(X) = -3u_0(1 - e^{-1/2}) \leq f(X)$, (iii) is proved.

From comparison principle, we have

$$w^-(X) \leq f(X) \leq w^+(X), \quad X \in B_{\sqrt{R^2+1}}(Y_0) \cap \Omega.$$

Therefore, the normal derivatives of $w^\pm$ and f satisfy

$$\frac{\partial w^-}{\partial \nu}(X_0) \leq \frac{\partial f}{\partial \nu}(X_0) \leq \frac{\partial w^+}{\partial \nu}(X_0),$$

which completes the proof. □

We define the first (Neumann) eigenvalue of self-adjoint operator $\mathcal{L}$ in complete self-shrinkers M^n in $\mathbb{R}^{n+1}$,

$$\lambda_1 = \inf_{f \in C^\infty(M)} \left\{ \int_M |\nabla f|^2 \rho; \quad \int_M f^2 \rho = 1, \int_M f \rho = 0 \right\}.$$

By (2.1), we have $\lambda_1 \leq \frac{1}{2}$. From the following lemma, λ_1 can be attained by the first eigenfunction u and $\lambda_1 > 0$ for any complete properly embedded self-shrinker.

Lemma 3.3. *Let M^n be a complete properly embedded self-shrinker in $\mathbb{R}^{n+1}$, then there exists a smooth function u satisfying $\int_M u^2 \rho = 1, \int_M u \rho = 0$ such that $\mathcal{L} u + \lambda_1 u = 0$ and $\int_M |\nabla u|^2 \rho = \lambda_1$.*

Proof. By the definition of λ_1, there exists a sequence $\{f_i\}$ satisfying

$$\int_M f_i^2\rho = 1, \int_M f_i\rho = 0 \quad \text{and} \quad \lim_{i\to\infty}\int_M |\nabla f_i|^2\rho = \lambda_1. \tag{3.5}$$

Since $\lambda_1 \leq 1/2$, then there exists a N_0 such that, for any $i \geq N_0$, $\int_M |\nabla f_i|^2\rho \leq 1$.

Define two Sobolev spaces $L^2(\Omega,\rho)$, $H^1(\Omega,\rho)$ for any set $\Omega \subset M$ by

$$\begin{aligned} L^2(\Omega,\rho) &= \{f\ ; \int_\Omega f^2\rho < \infty\ \} \\ H^1(\Omega,\rho) &= \{f\ ; \int_\Omega f^2\rho < \infty \text{ and } \int_\Omega |\nabla f|^2\rho < \infty\}. \end{aligned}$$

Since $H^1(D_r,\rho)$ is a Hilbert space, then there is a subsequence f_{n_i} converging to some $u_r \in H^1(D_r,\rho)$ weakly, and there is a subsequence $f_{n_{k_i}}$ converging to some $u_{r+1} \in H^1(D_{r+1},\rho)$ weakly and so on. Hence we can take the diagonal sequence, denoted by f_k for short, such that f_k converges to some $u_K \in H^1(K,\rho)$ weakly for any compact set $K \subset M$, i.e., we can define u on M such that $u\mid_K = u_K$. By compact embedding theorem, sequence f_k converges to u in the space $L^2(K,\rho)$ strongly for any compact set K, then

$$\int_K u^2\rho = \lim_{k\to\infty}\int_K f_k^2\rho \leq 1$$

By weak convergence in $H^1(D_r,\rho)$, we have

$$\begin{aligned} &\int_K (\nabla u\cdot\nabla f_k)\rho = \lim_{j\to\infty}\int_K (\nabla f_j\cdot\nabla f_k)\rho \\ \leq&\frac{1}{2}\lim_{j\to\infty}\int_K (|\nabla f_j|^2 + |\nabla f_k|^2)\rho \leq \frac{\lambda_1}{2} + \frac{1}{2}\int_K |\nabla f_k|^2\rho. \end{aligned}$$

Hence, we obtain

$$\int_M u^2\rho \leq 1, \qquad \int_M |\nabla u|^2\rho \leq \lambda_1. \tag{3.6}$$

For any sufficient small $\epsilon > 0$ and compact set $K \subset \mathbb{R}^{n+1}$, there exists a k such that $\int_K |u-f_k|^2\rho \leq \epsilon$, combining (3.5) and (3.6) and Cauchy inequality

$$\begin{aligned} |\int_K u\rho| \leq& \int_K |u-f_k|\rho + |\int_K f_k\rho| \leq \int_K |u-f_k|\rho + \int_{M\setminus K} |f_k|\rho \\ \leq&\sqrt{\int_K\rho\int_K |u-f_k|^2\rho} + \sqrt{\int_{M\setminus K}\rho\int_{M\setminus K}|f_k|^2\rho} \\ \leq&\sqrt{\epsilon\int_K\rho} + \sqrt{\int_{M\setminus K}\rho}. \end{aligned} \tag{3.7}$$

Since M has Euclidean volume growth, then (3.7) implies

$$\int_M u\rho = 0.$$

Now we prove $\int_M u^2\rho = 1$. By Logarithmic type Sobolev inequalities on self-shrinkers(see also [7])

$$\int_M |X|^2 f_k^2\rho \leq 16\int_M |\nabla f_k|^2\rho + 4n\int_M f_k^2\rho. \tag{3.8}$$

In fact, we multiply a smooth function g with compact support on the both sides of (2.2), and integral by parts,

$$\int_M (|X|^2-2n)g^2\rho = \int_M (\nabla|X|^2\cdot\nabla g^2)\rho \leq \frac{1}{2}\int_M |X|^2g^2\rho + 8\int_M |\nabla g|^2\rho.$$

If we approach f_k using such function g, we can get (3.8).

By (3.8), we have

$$r^2\int_{M\setminus D_r} f_k^2\rho \leq 16\int_M |\nabla f_k|^2\rho + 4n\int_M f_k^2\rho \leq 16+4n,$$

then

$$\int_{D_r} u^2\rho = \lim_{k\to\infty}\int_{D_r} f_k^2\rho = 1 - \lim_{k\to\infty}\int_{M\setminus D_r} f_k^2\rho \geq 1-\frac{16+4n}{r^2}. \tag{3.9}$$

Hence combining (3.6), $\int_M u^2\rho = 1$. By the definition of λ_1, we get $\int_M |\nabla u|^2\rho = \lambda_1$. Let us define a functional

$$I(f) = \int_M |\nabla f|^2\rho - 2\lambda_1\int_M fu\rho$$

and $\overline{f} = \int_M f\rho/\int_M\rho$, then

$$\begin{aligned} I(f) = &\int_M |\nabla f|^2\rho - 2\lambda_1\int_M (f-\overline{f})u\rho \geq \int_M |\nabla f|^2\rho - \lambda_1\int_M \left((f-\overline{f})^2+u^2\right)\rho \\ = &-\lambda_1 + \int_M |\nabla f|^2\rho - \lambda_1\int_M (f-\overline{f})^2\rho \geq -\lambda_1. \end{aligned}$$

Since $I(u) = -\lambda_1$, then the function u attains the minimum of $I(\cdot)$, hence $\frac{\partial}{\partial\epsilon}\mid_{\epsilon=0} I(u+\epsilon\varphi) = 0$ for any $\varphi\in C_c^\infty(M)$. By a simple computation, we have

$$\int_M (\mathcal{L}u+\lambda_1 u)\varphi\rho = 0.$$

By the regularity theory of elliptic equations(see [9] for example), we complete the Lemma. □

Let Σ be an n-dimensional manifold embedded in $\mathbb{R}^{n+1}$ (with or without boundary) satisfying two sides δ_0-ball condition(see Introduction for the definition). By a simple observation, $\Sigma\cap B_X(\frac{\delta_0}{|X|^2})$ has only one connect component for $X\in\Sigma$ with $|X|\geq 1$, where $B_X(\frac{\delta_0}{|X|^2})$ is a ball in $\mathbb{R}^{n+1}$ with radius $\frac{\delta_0}{|X|^2}$ at point X. In fact, if there is another

connect component Γ which don't through X, then there is no two sides δ_0-ball condition at any $Y \in \Gamma \cap B_X(\frac{\delta_0}{|X|^2})$.

Lemma 3.4. *Let Ω is a bounded domain in $\mathbb{R}^{n+1}$ with $\partial\Omega \in C^1$, and $\Sigma = \{X \in \partial\Omega : f(X) \neq 0\}$ for some function $f \in C^1(\overline{\Omega})$. If Σ has a δ_0-neighborhood of $\partial\Omega$ which satisfies two sides δ_0-ball condition and $\Sigma \cap B_1 = \emptyset$, then*

$$\int_{\partial\Omega} |f|^2\rho \leq C(n,\delta_0)d_0^2 \int_\Omega \left(|\overline{\nabla} f|^2 + |f|^2\right)\rho,$$

where $d_0 = \max\{dist(x,0) : x \in \Omega\}$, $C(n,\delta_0)$ is a constant depending only on n, δ_0.

Proof. Clearly, we can assume δ_0 is sufficiently small and $0 < \delta_0 < 1$. Firstly, we consider a simple case: if $B_{X_0}(\frac{\delta_1}{|X_0|^2}) \cap \partial\Omega$ belongs to a hyperplane $\{x_{n+1} = \delta_2\}$ for some $0 < \delta_1 < 1$ and $|\delta_2| \leq |X_0|$.

Let $\Gamma_1 = \partial\Omega \cap B_{X_0}(\frac{\delta_1}{2|X_0|^2})$, $B_1^+ = B_{X_0}(\frac{\delta_1}{|X_0|^2}) \cap \Omega$, $\zeta \in C_c^\infty(B_{X_0}(\frac{\delta_1}{|X_0|^2}))$, such that $\zeta\mid_{B_{X_0}(\frac{\delta_1}{|X_0|^2})} \geq 0$, $\zeta\mid_{B_{X_0}(\frac{\delta_1}{2|X_0|^2})} \equiv 1$ and $|\overline{\nabla}\zeta| \leq \frac{4|X_0|^2}{\delta_1}$. Denote $x' = (x_1, \cdots, x_n)$, we have

$$\begin{aligned}\int_{\Gamma_1} |f|^2\rho dx' \leq & \int_{\{x_{n+1}=0\}} \zeta f^2 \rho dx' = -\int_{B_1^+} (\zeta f^2\rho)_{x_{n+1}} dx \\ = & -\int_{B_1^+} \left(f^2\zeta_{x_{n+1}}\rho + 2ff_{x_{n+1}}\zeta\rho - \frac{x_{n+1}}{2} f^2\zeta\rho\right) dx \\ \leq & C(\delta_1)\int_{B_1^+} \left(|X_0|^2 f^2 + \frac{|\overline{\nabla} f|^2}{|X_0|^2}\right)\rho dx.\end{aligned} \tag{3.10}$$

For general case of $X_0 \in \{X \in \partial\Omega : \ f(X) \neq 0\}$ with $|X_0| \geq 1$, let $\Gamma = \partial\Omega \cap B_{X_0}(\frac{\delta_0}{2|X_0|^2})$ and $B^+ = B_{X_0}(\frac{\delta_0}{|X_0|^2}) \cap \Omega$. Now we use the standard technique of localization in PDE. In fact, we can assume the tangent plane at X_0 is horizontal, namely, $\{x_{n+1} = \delta_3\}$ for some $\delta_3 \in [-X_0, X_0]$. Then there is a function $\varphi \in C^1(\mathbb{R}^n)$ such that $B^+ = \{x \in B_{X_0}(\frac{\delta_0}{|X_0|^2});\ x_{n+1} > \varphi(x_1, \cdots, x_n)\}$. We define a transformation $\tau : x \to y$ by

$$\begin{cases} y_j = x_j \qquad \text{for} \quad 1 \leq j \leq n \\ y_{n+1} = x_{n+1} - \varphi(x_1, \cdots, x_n) + \delta_3 \end{cases},$$

By two sides δ_0-ball condition, we have $|\overline{\nabla}\varphi| \leq C(n,\delta_0)|X_0|^2$. Let $f(y) = f(\tau^{-1}(y))$ and $\tau(B^+)$ in (3.10), we have inequality

$$\int_\Gamma |f|^2\rho \leq C(n,\delta_0)|X_0|^2 \int_{B^+} \left(f^2 + |\overline{\nabla} f|^2\right)\rho dx. \tag{3.11}$$

For any $X_1 \in \Sigma$, let $B_1 = B_{X_1}(\frac{\delta_0}{|X_1|^2})$, then we choose $X_2 \in \Sigma \setminus \frac{1}{2}B_1$, here $\frac{1}{2}B_1 = B_{X_1}(\frac{\delta_0}{2|X_1|^2})$ and let $B_2 = B_{X_2}(\frac{\delta_0}{|X_2|^2})$, and we choose $X_3 \in \Sigma \setminus \bigcup_{i=1}^{2} \frac{1}{2}B_i$. If we have chosen $X_1, \cdots, X_k$, then we choose $X_{k+1} \in \Sigma \setminus \bigcup_{i=1}^{k} \frac{1}{2}B_i$. Since $\overline{\Sigma}$ is compact, then $\Sigma \subset \bigcup_{i=1}^{l} \frac{1}{2}B_i$ for some finite number l.

For such balls we have chosen, we claim

$$\sum_k \chi_{B_k}(X) \le C(n, \delta_0) \text{ for } X \in \partial\Omega. \tag{3.12}$$

If $X \in B_k$ with $|X| \ge 3\max\{1, \delta_0\}$, then $|X - X_k| \le \frac{\delta_0}{|X_k|^2}$. Since $X_k \in \mathbb{R}^{n+1} \setminus B_1$ and $0 < \delta_0 < 1$, then we deduce $|X - X_k| \le 1$ and

$$|X - X_k| \le \frac{\delta_0}{|X - (X - X_k)|^2} \le \frac{\delta_0}{(|X| - |X - X_k|)^2} \le \frac{9\delta_0}{4|X|^2}.$$

That is $X_k \in B_X(\frac{9\delta_0}{4|X|^2})$, which implies the claim immediately.

Hence combining (3.11) and (3.12), we have

$$\begin{aligned}\int_{\partial\Omega} |f|^2\rho \le & \sum_k \int_{\frac{1}{2}B_k \cap \Sigma} |f|^2\rho \le C(n, \delta_0) d_0^2 \sum_k \int_{B_k \cap \Omega} \left(f^2 + |\overline{\nabla} f|^2\right)\rho \\ \le & C(n, \delta_0) d_0^2 \int_{\Omega} \left(f^2 + |\overline{\nabla} f|^2\right)\rho.\end{aligned}$$

□

Now, we give a positive lower bound of λ_1 for compact embedded self shrinkers, or a complete embedded self shrinkers with two sides δ_0-ball condition.

Theorem 3.5. *Let M^n is a a compact embedded self-shrinker in $\mathbb{R}^{n+1}$, then the first eigenvalue λ_1 of operator $\mathcal{L}$ on M satisfies: $\lambda_1 \in (\frac{1}{4}, \frac{1}{2}]$.*

Proof. Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with $\partial\Omega = M \cup S_R$, here S_R is n-sphere with sufficient large radius R and centered at the origin. We consider Dirichlet problem

$$\begin{cases} \overline{\mathcal{L}} f = 0 & \text{in} \quad \Omega \\ f\,|_M = u\ ,\ f\,|_{S_R} = 0\ , \end{cases}$$

where u is the first eigenfunction of self-adjoint operator $\mathcal{L}$ in M, i.e., $\mathcal{L}u + \lambda_1 u = 0$ and satisfying $\int_M u^2\rho = 1$. By Lemma 3.2, we get $|\overline{\nabla} f(X_0)| \le 3\max_{X \in M}|u(X)|R$ for any $X_0 \in S_R$. Since

$$\int_M f_\nu \mathcal{L}u\rho = -\lambda_1 \int_M f_\nu u\rho = -\lambda_1 \int_\Omega \overline{\mathrm{div}}(\rho f \overline{\nabla} f) = -\lambda_1 \int_\Omega |\overline{\nabla} f|^2\rho, \tag{3.13}$$

and combining (1.1), Theorem 3.1 and (3.13), we have

(3.14)
$$\begin{aligned}0\geq&\int_\Omega|\overline{\nabla}^2f|^2\rho+\frac{1}{2}\int_\Omega|\overline{\nabla}f|^2\rho-2\lambda_1\int_\Omega|\overline{\nabla}f|^2\rho-\int_M h(\nabla u,\nabla u)\rho-\int_{S_R}f_\nu^2\frac{R}{2}\rho\\ \geq&\int_\Omega|\overline{\nabla}^2f|^2\rho+(\frac{1}{2}-2\lambda_1)\int_\Omega|\overline{\nabla}f|^2\rho-\int_M h(\nabla u,\nabla u)\rho-\frac{9}{2}\max_{X\in M}|u(X)|^2R^3\int_{S_R}\rho.\end{aligned}$$

We can assume $\int_M h(\nabla u,\nabla u)\rho\leq 0$, or else we consider the bounded domain U with $\partial U=M$ instead of Ω. Since $|\overline{\nabla}^2 f|$ can not be zero everywhere, we let R go to infinite in (3.14) and get $\lambda_1>\frac{1}{4}$. □

For noncompact case, we have following result.

Theorem 3.6. *Let M^n is a complete noncompact embedded self-shrinker with two sides δ_0-ball condition in $\mathbb{R}^{n+1}$, then the first eigenvalue λ_1 of operator $\mathcal{L}$ on M satisfying: $\lambda_1\in[\frac{1}{4},\frac{1}{2}]$.*

Proof. Assume $\lambda_1<\frac{1}{4}$, and we'll deduce contradiction.

For sufficient large r, let a radial cut-off function $\eta\in C_c^2(B_{\frac{2r}{3}})$, such that $\eta\mid_{B_{\frac{2r}{3}}}\geq 0$, $\eta\mid_{B_{\frac{r}{3}}}\equiv 1$, $|\eta'|\leq\frac{C}{r}$ and $|\eta''|\leq\frac{C}{r^2}$. Let $\partial B_r=S_1\cup S_2$ and $\partial S_1=\partial S_2=M\cap\partial B_r$. There exist C^3-manifolds M_r^1 and M_r^2 approach $D_r\cup S_1$ and $D_r\cup S_2$, respectively, and $D_r\subset M_r^1\cap M_r^2$, $\frac{1}{2}\langle X,\nu\rangle+H|_X\leq 0$ for any $X\in M_r^i$, where ν points in the domain enclosed by $X\in M_r^i$ and $i=1,2$.(see appendix A for the proof).

Let $\Omega=\Omega(R,r)$ is bounded open set enclosed by M_r^1 and ∂B_R for $R\gg r$, u is the first eigenfunction for λ_1 satisfying $\int_M u^2\rho=1$. Without loss of generality, we suppose that there exists a sequence $\{r_i\}$ go to infinite such that

$$\lim_{r_i\to\infty}\int_{D_{r_i}}h(\nabla(\eta u),\nabla(\eta u))\rho\leq 0,$$

otherwise, we consider another domain enclosed by M_r^2 and ∂B_R instead of Ω.

Now we consider Dirichlet boundary value problem

$$\begin{cases}\overline{\mathcal{L}}f=0 & \text{in}\quad\Omega\\ f=0 & \text{on}\quad\partial B_R\cup(M_r^1\setminus D_r)\\ f=\eta(|X|)u & \text{on}\quad D_r,\end{cases}$$

then by Theorem 3.1, we have

$$
\begin{aligned}
0 = &\int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho + 2\int_{D_r} f_\nu \mathcal{L}(\eta u)\rho - \int_{D_r} h(\nabla(\eta u), \nabla(\eta u))\rho \\
&- \int_{\partial B_R} f_\nu^2 (\frac{\langle X, \nu\rangle}{2} + H)\rho - \int_{M_r^1} f_\nu^2 (\frac{\langle X, \nu\rangle}{2} + H)\rho,
\end{aligned}
\tag{3.15}
$$

where the vector ν points out of Ω in the last term of the above equality. Since

$$
\mathcal{L}\eta = \eta' \left(\frac{n}{|X|} - \frac{|X|}{2} - \frac{|X^T|^2}{|X|^3} \right) + \eta'' \frac{|X^T|^2}{|X|^2},
$$

then $|\mathcal{L}\eta| \leq C(n)$, combining $\mathcal{L}u + \lambda_1 u = 0$, for any $\epsilon > 0$,

$$
\begin{aligned}
\int_{D_r} f_\nu \mathcal{L}(\eta u)\rho = &\int_{D_r} f_\nu (\eta \mathcal{L} u + 2\nabla\eta\cdot\nabla u + u\mathcal{L}\eta)\rho \\
\geq & -\lambda_1 \int_{D_r} f f_\nu \rho + \int_{D_{2r/3}\setminus D_{r/3}} (f_\nu \eta' \frac{X^T}{|X|}\cdot\nabla u - C(n)|f_\nu|\cdot|u|)\rho \\
\geq & -\lambda_1 \int_\Omega \overline{\mathrm{div}}(\rho f \overline{\nabla} f) - C(n)\int_{D_{2r/3}\setminus D_{r/3}} |f_\nu|(\frac{|\nabla u|}{r} + |u|)\rho \\
\geq & -\lambda_1 \int_\Omega |\overline{\nabla} f|^2\rho - C(n)\int_{D_{2r/3}\setminus D_{r/3}} \left(\frac{1}{\epsilon}\frac{|f_\nu|^2}{r^2} + \epsilon|\nabla u|^2 + \epsilon r^2 |u|^2\right)\rho.
\end{aligned}
\tag{3.16}
$$

By Lemma 3.4,

$$
\int_{D_{2r/3}\setminus D_{r/3}} \frac{|f_\nu|^2}{r^2}\rho \leq C(n,\delta_0)\int_{\Omega\cap(B_{5r/6}\setminus B_{r/6})} \left(|\overline{\nabla}^2 f|^2 + |\overline{\nabla} f|^2\right)\rho.
\tag{3.17}
$$

By the definition of u and (3.9),

$$
\int_{M\setminus D_r} u^2\rho \leq \frac{16+4n}{r^2}.
\tag{3.18}
$$

Since $\frac{1}{2}\langle X, \nu\rangle + H \leq 0$ in M_r^1, then combining (3.15)$\sim$(3.18) and Lemma 3.2, we get

$$
\begin{aligned}
0 \geq &\int_\Omega |\overline{\nabla}^2 f|^2\rho + \frac{1}{2}\int_\Omega |\overline{\nabla} f|^2\rho - \int_{D_r} h(\nabla(\eta u), \nabla(\eta u))\rho - \int_{\partial B_R} f_\nu^2 \frac{R}{2}\rho \\
&- 2\lambda_1 \int_\Omega |\overline{\nabla} f|^2\rho - C(n)\int_{D_{2r/3}\setminus D_{r/3}} \left(\frac{1}{\epsilon}\frac{|f_\nu|^2}{r^2} + \epsilon|\nabla u|^2 + \epsilon r^2|u|^2\right)\rho \\
\geq &\int_\Omega |\overline{\nabla}^2 f|^2\rho + (\frac{1}{2} - 2\lambda_1)\int_\Omega |\overline{\nabla} f|^2\rho - \int_{D_r} h(\nabla(\eta u), \nabla(\eta u))\rho \\
&- \frac{9}{2}\max_{X\in B_r} |u(X)|^2 R^3 \int_{\partial B_R}\rho - \frac{C(n,\delta_0)}{\epsilon}\int_{\Omega\cap(B_{5r/6}\setminus B_{r/6})} \left(|\overline{\nabla}^2 f|^2 + |\overline{\nabla} f|^2\right)\rho - C(n)\epsilon.
\end{aligned}
\tag{3.19}
$$

If $\lim_{R\to\infty}\int_{\Omega(R,r)} |\overline{\nabla}^2 f|^2\rho = +\infty$ or $\lim_{R\to\infty}\int_{\Omega(R,r)} |\overline{\nabla} f|^2\rho = +\infty$, then we let ϵ be sufficiently large such that $\frac{1}{2} - 2\lambda_1 - \frac{C(n,\delta_0)}{\epsilon} > 0$, and we find this is impossible by (3.19). Now

let

$$\epsilon = \epsilon(r,R) = \sqrt{\int_{\Omega\cap(B_{5r/6}\setminus B_{r/6})} \left(|\overline{\nabla}^2 f|^2 + |\overline{\nabla} f|^2\right)\rho},$$

then there exists a sequence $\{R_i\}$ and $\{r_i\}$ go to $+\infty$ such that

$$\lim_{i\to\infty} \epsilon(r_i,R_i) = \lim_{i\to\infty}\left(\max_{X\in B_{r_i}} |u(X)|^2 R_i^3 \int_{\partial B_{R_i}} \rho\right) = 0.$$

Hence by (3.19),

$$0 \geq \lim_{i\to\infty}\int_{\Omega(R_i,r_i)} |\overline{\nabla}^2 f|^2\rho + (\frac{1}{2} - 2\lambda_1)\lim_{i\to\infty}\int_{\Omega(R_i,r_i)} |\overline{\nabla} f|^2\rho. \tag{3.20}$$

Noting that u is a smooth function satisfying $\mathcal{L}u + \lambda_1 u = 0$ and Lemma 3.4, we have

$$0 < \int_{D_{r_0}} |\nabla u|^2\rho \leq C(n,\delta_0) r_0^2 \int_{\Omega\cap B_{2r_0}} \left(|\overline{\nabla}^2 f|^2 + |\overline{\nabla} f|^2\right)\rho, \tag{3.21}$$

for some fixed $r_0 > 1$. Then (3.20) is impossible. Therefore $\lambda_1 \geq \frac{1}{4}$. □

Corollary 3.7. *Let M^n is a complete embedded self-shrinker in $\mathbb{R}^{n+1}$ satisfying two sides δ_0-ball condition, and $\int_M f^2\rho < \infty$, then there is a Poincaré inequality*

$$\int_M (f - \overline{f})^2\rho \leq 4\int_M |\nabla f|^2\rho,$$

where $\overline{f} = \int_M f\rho / \int_M \rho$.

Proof. If f is not a constant, let $g = (\int_M f^2\rho - \overline{f}^2 \cdot \int_M \rho)^{-1/2}(f - \overline{f})$, then $\int_M g\rho = 0$ and $\int_M g^2\rho = 1$. Since λ_1 is the first eigenvalue of self-adjoint operator $\mathcal{L}$, combining the Theorem 3.5 and 3.6, we complete the proof. □

Now, let us recall a classical result of P. Yang and S.T. Yau [18].

Theorem 3.8. *(Yang-Yau) Let (Σ_g^2, ds^2) is an orientable Riemann surface of genus g with area $Area(\Sigma_g)$. Then we have*

$$\lambda_1(\Sigma_g) \leq \frac{8\pi(1+g)}{Area(\Sigma_g)},$$

where $\lambda_1(\Sigma_g)$ is the first eigenvalue of Δ on Σ_g.

Let $N^3 = (\mathbb{R}^3, (\rho\delta_{ij}))$ and M^2 is a compact embedded self-shrinker in $\mathbb{R}^3$. Let $\widetilde{g} = \widetilde{g}_{ij} d\theta_i d\theta_j$, $\widetilde{\nabla}$ and $\widetilde{\Delta}$ are the metric, the Levi-Civita connection and the Laplacian of M induced from N^3, respectively. Denote the self-shrinker M with metric $\widetilde{g}$ by $\widetilde{M}$. Let

$g_{ij}d\theta_i d\theta_j$ and $d\mu$ are the metric and the volume element of M induced from $\mathbb{R}^3$, then $\widetilde{g}_{ij} = \rho g_{ij}$. Denote the first eigenvalue of $\widetilde{M}$ by $\lambda_1(\widetilde{M})$, then

$$\begin{aligned}\lambda_1(\widetilde{M}) &= \inf_{\int_M f\rho=0} \frac{\int_M |\widetilde{\nabla} f|^2 \rho d\mu}{\int_M f^2 \rho d\mu} = \inf_{\int_M f\rho=0} \frac{\int_M \widetilde{g}^{ij} f_i f_j \rho d\mu}{\int_M f^2\rho d\mu} \\ &\geq \inf_{\int_M f\rho=0} \frac{\int_M g^{ij} f_i f_j \rho d\mu}{\int_M f^2 \rho d\mu} = \lambda_1 > \frac{1}{4},\end{aligned} \tag{3.22}$$

where we have used Theorem 3.5 in the last inequality in the above inequality.

Corollary 3.9. *Let M is a compact embedded self-shrinker in $\mathbb{R}^3$ with genus g, then*

$$\int_M \rho < 32\pi(1+g),$$

moreover,

$$\int_{D_r} 1 d\mu \leq 32 e^{\frac{1}{4}} \pi (1+g) r^2, \quad \text{for} \quad r \geq 1.$$

Proof. Combining (3.22) and Theorem 3.8, we get

$$\int_M \rho < 32\pi(1+g). \tag{3.23}$$

By (2.7) and (3.23) for $r \geq 1$, we have

$$\frac{1}{r^2} e^{-\frac{1}{4}} \int_{D_r} 1 d\mu \leq \frac{1}{r^2} \int_{D_r} e^{-\frac{|X|^2}{4r^2}} d\mu \leq \int_{D_r} e^{-\frac{|X|^2}{4}} d\mu \leq 32\pi(1+g), \tag{3.24}$$

which implies

$$\int_{D_r} 1 d\mu \leq 32 e^{\frac{1}{4}} \pi (1+g) r^2, \quad \text{for} \quad r \geq 1. \tag{3.25}$$

□

For a non-negative integer g and a constant $D > 0$, let $S_{g,D}$ denote the space of all compact embedded self-shrinkers in $\mathbb{R}^3$ with genus at most g, and diameter at most D. Now, we are in a position to prove a compactness theorem.

Theorem 3.10. *For each fixed g and D, the space $S_{g,D}$ is compact. Namely, any sequence has a subsequence that converges uniformly in the C^k topology (for any $k \geq 0$) to a surface in $S_{g,D}$.*

Proof. For any compact surface Σ in $\mathbb{R}^3$, by Gauss-Bonnet formula, we have

$$\int_\Sigma |B|^2 = \int_\Sigma H^2 - 2\int_\Sigma K = \int_\Sigma H^2 - 4\pi\chi(\Sigma), \tag{3.26}$$

where $\chi(\Sigma)$ is the Euler number of surface Σ. If $\Sigma \in S_{g,D}$, then by (2.8), there is a $X \in \Sigma$ such that $|X| = \sqrt{2n}$. We obtain $\Sigma \subset B_{D+\sqrt{2n}}$, then from (3.25), we have area estimate

$$\int_\Sigma 1 d\mu \le 32 e^{\frac{1}{4}} \pi (1+g)(D+\sqrt{2n})^2. \tag{3.27}$$

By (2.3), (3.26) and (3.27), we have

$$\int_\Sigma |B|^2 = \int_\Sigma 1 d\mu - 4\pi\chi(\Sigma) \le 32 e^{\frac{1}{4}} \pi (1+g)(D+\sqrt{2n})^2 + 8\pi(g-1). \tag{3.28}$$

By Proposition 5.10 of [6](see also [1] and [5]), we complete the proof. □

4. Appendix A Construction of Smooth Hypersurface with Nonpositive Mean Curvature

We follow the method of W.W.Meeks, III and S.T.Yau in [12] to smooth out $\partial D_r \cap S_1$ of $D_r \cup S_1$ by induction on a number of edges of $D_r \cup S_1$. Let Γ be a connected component of $M^n \cap \partial B_r$, and we can assume Γ is a $n-1$-manifold (Or else, we may deform the S_1 to $\widetilde{S_1}$ such that mean curvature of $\widetilde{S_1}$ is positive under the metric $(e^{-\frac{|X|^2}{2n}}\delta_{ij})$, and $\widetilde{S_1} \cap M$ may be empty in the neighborhood of Γ). For any $X \in \Gamma$, there exist a neighborhood $U \ni X$ and an isometry $\phi : (U, (e^{-\frac{|X|^2}{2n}}\delta_{ij})) \to (\phi(U), (g_{ij}))$, such that $\phi(\Gamma \cap U)$ is a domain in $\{x_1 = x_{n+1} = 0\}$ plane, $\phi(M \cap U)$ is a domain in $\{x_{n+1} = 0\}$ plane, and $\phi(S_1 \cap U)$ is a domain in $\{x_{n+1} = x_1\}$ plane.

If f is any function of x_1, then the mean curvature of the surface defined by f is positive with respect to the downward direction if and only if

$$g^{11} f'' + \sum_{i=0}^{p} a_i (f')^i > 0, \tag{4.1}$$

here p is an integer depending on n, a_i is a function in $\phi(U) \cap \{x_{n+1} = 0\}$ depending on g_{ij}, its first derivatives and f', and if f' is bounded, then a_i is bounded. Since the mean curvature of $f = 0$ is zero, then (4.1) implies $a_0 = 0$. We can find a function f satisfies

$$f^{(i)}(0) = 0 \ for \ i = 0, 1, 2, 3, \ and \ \frac{f'(t)}{f''(t)} \to Ct (t \to 0).$$

Then there exists $\epsilon > 0$ such that (4.1) holds in $0 < t < \epsilon$ and $f'(t) = t$ in $\epsilon \le t \le \epsilon_0$ for some $\epsilon_0 > \epsilon$. We can smooth out f' and integrate to obtain the required f satisfying (4.1).

Since Γ is compact, then there are finite neighborhoods to cover Γ and finite $\{\epsilon_i\}$ which associate to such neighborhoods. We choose the smallest ϵ in $\{\epsilon_i\}$ to construct required f such that the mean curvature of the surface defined by f is positive with respect to the downward direction. Then we deform other components of $M \cap S_1$ to complete the proof.

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